An almost sure limit theorem for super-Brownian motion [1]

Li Wang[2]

Abstract

We establish an almost sure scaling limit theorem for super-Brownian motion on $\mathbb{R}^d$ associated with the semi-linear equation $u_t = \frac{1}{2}\Delta u + \beta u - \alpha u^2$, where α and β are positive constants. In this case, the spectral theoretical assumptions that required in Chen et al (2008) are not satisfied. An example is given to show that the main results also hold for some sub-domains in $\mathbb{R}^d$.

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1 Introduction

Let $B_b(\mathbb{R}^d)$ (respectively, $B_b^+(\mathbb{R}^d)$) be the set of all bounded (respectively, non-negative) Borel measurable functions on $\mathbb{R}^d$. Denote by $C_b(\mathbb{R}^d)$ the space of bounded continuous functions on R^d. Let $C^{k,\eta}(\mathbb{R}^d)$ denote the space of Hölder continuous functions of index $\eta \in (0,1]$ which have derivatives of order k, and set $C^{\eta}(\mathbb{R}^d) := C^{0,\eta}(\mathbb{R}^d)$. Write $C_b^1(\mathbb{R}^d)$ for the space of bounded functions in $C^{1,1}(\mathbb{R}^d)$. Let L be an elliptic operator on $\mathbb{R}^d$ of the form

$$L := \frac{1}{2}\nabla \cdot A\nabla + B \cdot \nabla,$$

where the matrix $A(x) = (a_{i,j}(x))$ is symmetric and positive definite for all $x \in \mathbb{R}^d$ with $a_{i,j}(x) \in C^{1,\eta}(\mathbb{R}^d)$ and $B(x) = (b_1(x), \cdots, b_d(x))$ is an $\mathbb{R}^d$-valued function with $b_i(x) \in C^{1,\eta}(\mathbb{R}^d)$, $i, j = 1, \cdots, d$. In addition, let $\alpha, \beta \in C^{\eta}(\mathbb{R}^d)$, and assume that α is positive, and β is bounded from above.

Let $\{X_t, t \geq 0\}$ be a super-diffusion corresponding to the operator $Lu + \beta u - \alpha u^2$ on $\mathbb{R}^d$. Denote by λ_c the generalized principal eigenvalue for the operator $L + \beta$ on $\mathbb{R}^d$, i.e.,

$$\lambda_c = \inf\{\lambda \in \mathbb{R} : L + \beta - \lambda \ \text{ posesses a Green's function}\}.$$

Let $\tilde{L}$ be the formal adjoint of L, the eigenfunction of $L+\beta$ corresponding to λ_c will be denoted by ϕ, and the eigenfunction of $\tilde{L}+\beta$ corresponding to λ_c will be denoted by $\tilde{\phi}$. The operator $L + \beta - \lambda_c$ is called product-critical if $\phi > 0$, ϕ and $\tilde{\phi}$ satisfy $\langle \phi, \tilde{\phi} \rangle < \infty$. In this case we normalize them by $\langle \phi, \tilde{\phi} \rangle = 1$. Engländer and Turaev (2002) proved that if $\lambda_c > 0$, $L + \beta - \lambda_c$

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[2]Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R. China.
E-mail: lwang@mail.bnu.edu.cn

is product-critical, $\alpha\phi$ is bounded and the initial state μ is such that $\langle \mu, \phi\rangle < \infty$, then for every positive continuous function f with compact support,

$$\lim_{t\to\infty} e^{-\lambda_c t}\langle X_t, f\rangle = N_\mu \langle \tilde{\phi}, f\rangle \quad \text{in distribution,}$$

where the limiting non-negative non-degenerate random variable N_μ was identified with the help of a certain invariant curve. Engländer and Winter (2006) improved the above result to show that the above convergence holds in probability.

Chen et al (2008) established that the above convergence in probability result holds for a large class of Dawson-Watanabe superprocesses. Moreover, if the following assumptions hold: (1) The underlying spatial motion ξ is either a symmetric Lévy process in $\mathbb{R}^d$ whose Lévy exponent $\Psi(\eta)$ is bounded from below by $c|\eta|^\alpha$ for some $c > 0$ and $\alpha \in (0, 2)$ when $|\eta|$ is large (we also denote its infinitesimal generator by L); or a symmetric diffusion on $\mathbb{R}^d$ with infinitesimal generator

$$L = \rho(x)^{-1}\nabla \cdot (\rho A \nabla) \tag{1.1}$$

where $A(x) = (a_{i,j}(x))$ is uniformly elliptic and bounded with $a_{i,j} \in C^1_b(\mathbb{R}^d)$ and the function $\rho(x) \in C^1_b(\mathbb{R}^d)$ is bounded between two positive constants; (2) $\beta \in K_\infty(\xi) \cap C_b(\mathbb{R}^d)$ and $\alpha \in K_\infty(\xi) \cap B^+_b(\mathbb{R}^d)$; (3) $\lambda_1 := \lambda_1(\beta) < 0$ ($\lambda_1(\beta)$ is the smallest spectrum of $L + \beta$, and $K_\infty(\xi)$ is the space of Green tight-functions for ξ), then for every bounded measurable function f on $\mathbb{R}^d$ with compact support whose set of discontinuous points has zero m measure,

$$\lim_{t\to\infty} e^{\lambda_1 t}\langle X_t, f\rangle = M^\phi_\infty \int_{\mathbb{R}^d} f(x)\phi(x)m(dx), \quad \mathbb{P}_{\delta_x} - a.s.$$

where ϕ is the normalized positive eigenfunction of $L+\beta$ corresponding to λ_1, M^ϕ_∞ is the almost sure limit of $M^\phi_t := e^{\lambda_1 t}\langle X_t, \phi\rangle$ and m is the measure with respect to which the underlying spatial motion is symmetric.

Note that if L is of the form (1.1), then the generalized principal value λ_c for operator $L + \beta$ on $\mathbb{R}^d$ equals to $-\lambda_1(\beta)$. The assumptions (1), (2) and (3) above were used to guarantee that the associated Schrödinger operator $L + \beta$ has a spectral gap and has an L^2-eigenfunction corresponding to λ_1.

In this paper, we consider the supercritical super-Brownian motion on $\mathbb{R}^d$, $d \geq 1$, corresponding to the operator $\frac{1}{2}\Delta u + \beta u - \alpha u^2$, where α, β are positive constants. Thus $\beta \notin K_\infty(\xi)$ (here, ξ is Brownian motion), $\lambda_c = \beta$ and

$$\frac{1}{2}\Delta + \beta - \lambda_c = \frac{1}{2}\Delta$$

has normalized eigenfunction $\phi = 1$, which is not L^2-integrable. Therefore this case was not included in the setup of the above papers. On the other hand, the corresponding almost sure limit theorem is known for *discrete particle systems*.

Using techniques from Fourier transform theory, Watanabe (1967) proved an almost sure limit theorem for branching Brownian motion in $\mathbb{R}^d$ and in certain sub-domains in it. However, the proof in Watanabe (1967) is thought to have a gap as expressed in Engländer (2008). In this paper, using the main idea from Watanabe (1967), we prove an almost sure limit theorem for the super-Brownian motion and fill this gap in Proposition 3.1 and 3.2. In the more general case where α, $\beta \in C^\eta(\mathbb{R}^d)$, α is bounded and positive, and β is compactly supported, we can immediately apply the result in Chen et al (2008) to give an almost sure limit theorem.

The remainder of the paper is organized as follows. In section 2 we give some preliminary results about super-Brownian motion. The main results and the corresponding proofs are presented in section 3. However, in order to facilitate the understanding of readers, we first give some of the basic results, which lead to our main theorems. In section 4 we will give an example to show that the main results also hold for some sub-domains in $\mathbb{R}^d$.

2 Super-Brownian Motion

Let $M_F(\mathbb{R}^d)$ be the set of finite measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. Let $M_c(\mathbb{R}^d)$ be the subset of all compactly supported measures. The space of continuous functions with compact support (respectively, non-negative) will be denoted by $C_c(\mathbb{R}^d)$ (resp. $C_c^+(\mathbb{R}^d)$). Let $C_c^\eta(\mathbb{R}^d)$ denote the space of functions in $C^\eta(\mathbb{R}^d)$ with compact support. Denote by λx the inner product for λ, $x \in \mathbb{R}^d$. Denote by $|\cdot|$ the Euclidean norm.

Let $\xi = (\Omega, \xi_t, \mathscr{F}, \mathscr{F}_t, \mathbf{P}_x)$ be a Brownian motion on $\mathbb{R}^d$ with transition semigroup $\{P_t, t \geq 0\}$. Suppose $X = \{W, \mathscr{G}, \mathscr{G}_t, X_t, \mathbb{P}_\mu, \mu \in M_F(\mathbb{R}^d)\}$ is a time-homogeneous càdlàg super-Markov process corresponding to the operator $\frac{1}{2}\Delta u + \beta u - \alpha u^2$ where $\alpha, \beta \in C^\eta(\mathbb{R}^d)$. More precisely, X is a super-Brown motion with $X_t \in M_F(\mathbb{R}^d)$, $t \geq 0$, and the Laplace functional

$$\mathbb{P}_\mu \left[\exp(\langle -f, X_t\rangle)\right] = \exp(\langle -u(t,\cdot), \mu\rangle)$$

with $\mu \in M_F(\mathbb{R}^d)$, $f \in B_b^+(\mathbb{R}^d)$, where u is the unique solution of the integral equation

$$u(t,x) + \int_0^t ds \int_E \alpha(y) u(s,y)^2 P_{t-s}^\beta(x,dy) = P_t^\beta f(x),$$

where $P_t^\beta f(x) := \mathbf{P}_x[e^{\int_0^t \beta(\xi_s)ds} f(\xi_t)]$. As usual, $\langle f, \mu\rangle$ denotes the integral $\int_{\mathbb{R}^d} f(x)\mu(dx)$. The first two moments for X_t are given as follows: for every $f \in B_b^+(\mathbb{R}^d)$ and $t \geq 0$,

$$\mathbb{P}_\mu[\langle f, X_t\rangle] = \mu(P_t^\beta f), \tag{2.1}$$

$$\mathbb{P}_\mu[\langle f, X_t\rangle^2] = \mu(P_t^\beta f)^2 + 2\int_0^t ds \int_{\mathbb{R}^d} \alpha(y)(P_s^\beta f(y))^2 \mu P_{t-s}^\beta(dy). \tag{2.2}$$

For the definition of super processes in general, the reader is referred to Dynkin (1991, 2002) or Dawson (1993), and for more of the definition in the particular setting above, see Engländer and Pinsky (1999).

In the sequel, we will assume that α and β are positive constants unless otherwise specified. Let $\varphi_\lambda(x) := e^{i\lambda x}$. For $f \in C_c(\mathbb{R}^d)$, denote its Fourier transform by $\widehat{f}(\lambda) = \int_{\mathbb{R}^d} f(x)\varphi_\lambda(x)dx$. Then $\widehat{f}(\lambda)$ is continuous and

$$P_t\varphi_\lambda(x) = (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \varphi_\lambda(y) \exp\left\{-\frac{|y-x|^2}{2t}\right\} dy = \varphi_\lambda(x)\exp\{-\frac{1}{2}|\lambda|^2 t\} := \widehat{P}_t(x,\cdot)(\lambda). \tag{2.3}$$

Denote the transition density of P_t by $p_t(x,y)$, and let $\rho(\lambda) := \beta - \frac{1}{2}|\lambda|^2$. Then

$$\begin{aligned}\mathbb{P}_{\delta_x}[\langle f, X_t\rangle] &= P_t^\beta f(x) = e^{\beta t}\int_{\mathbb{R}^d} f(y)p_t(x,y)dy \\ &= e^{\beta t}\int_{\mathbb{R}^d} f(y)p_t(y,x)dy\end{aligned}$$

$$
\begin{aligned}
&= e^{\beta t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\widehat{P}_t(y,\cdot)(\lambda)\overline{\varphi_\lambda(x)}\frac{d\lambda}{(2\pi)^d}dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\rho(\lambda)t} f(y)\varphi_\lambda(y)\overline{\varphi_\lambda(x)}dy\frac{d\lambda}{(2\pi)^d} \\
&= \int_{\mathbb{R}^d} e^{\rho(\lambda)t}\widehat{f}(\lambda)\overline{\varphi_\lambda(x)}\frac{d\lambda}{(2\pi)^d}.
\end{aligned}
$$

For $f \in C_c(\mathbb{R}^d)$, let $g_f(x) := \mathbb{P}_{\delta_x}[\langle f, X_t\rangle] = P_t^\beta f(x)$. We will write $g(x)$ for $g_f(x)$ if there is no ambiguity. Then

$$
\begin{aligned}
\widehat{g}(\lambda) &= \int_{\mathbb{R}^d} P_t^\beta f(x)\varphi_\lambda(x)dx \\
&= e^{\beta t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)P_t(x,dy)\varphi_\lambda(x)dx \\
&= e^{\beta t}\int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d}\varphi_\lambda(x)P_t(y,dx)dy \\
&= e^{\rho(\lambda)t}\int_{\mathbb{R}^d} f(y)\varphi_\lambda(y)dy \\
&= e^{\rho(\lambda)t}\widehat{f}(\lambda).
\end{aligned}
$$

Hence,

$$
g(x) = \int_{\mathbb{R}^d}\widehat{g}(\lambda)\overline{\varphi_\lambda(x)}\frac{d\lambda}{(2\pi)^d}. \tag{2.4}
$$

Note that for each $t > 0$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} e^{-\frac{t}{2}\rho(\lambda)}|\widehat{g}(\lambda)|d\lambda &= \int_{\mathbb{R}^d} e^{\frac{t}{2}\rho(\lambda)}|\widehat{f}(\lambda)|d\lambda \\
&\le e^{\frac{\beta}{2}t}\int_{\mathbb{R}^d} e^{-\frac{1}{4}|\lambda|^2 t}d\lambda\int_{\mathbb{R}^d}|f(x)|dx \\
&= e^{\frac{\beta}{2}t}\left(\frac{2\sqrt{\pi}}{\sqrt{t}}\right)^d\int_{\mathbb{R}^d}|f(x)|dx < \infty. \qquad (2.5)
\end{aligned}
$$

Let $\mathscr{A} := \Bigg\{ f \in L^2(\mathbb{R}^d)$: f is continuous, $\widehat{f}(\lambda)$ exists and continuous in λ,

$$
f(x) = \int_{\mathbb{R}^d}\widehat{f}(\lambda)\overline{\varphi_\lambda(x)}\frac{d\lambda}{(2\pi)^d}
$$

and

$$
\int_{\mathbb{R}^d} e^{-\varepsilon\rho(\lambda)}|\widehat{f}(\lambda)|\frac{d\lambda}{(2\pi)^d} < \infty \text{ for some } \varepsilon > 0\Bigg\}.
$$

By (2.4) and (2.5), $\big\{g(x) = \mathbb{P}_{\delta_x}[\langle f, X_t\rangle]; f \in C_c(\mathbb{R}^d), t > 0\big\} \subset \mathscr{A}$.

The next lemma is a version of Lemma 3.1 in Watanabe (1967) and the proof is similar, so we omit the proof here.

Lemma 2.1 *For every $f \in C_c^+(\mathbb{R}^d)$ and every $\varepsilon > 0$, there exist f_1, $f_2 \in \mathscr{A}$ such that $\int_{\mathbb{R}^d}(f_2 - f_1)dx < \varepsilon$.*

3 Limit Theorems

If $f \in \mathscr{A}$, then $\langle X_t, f\rangle = \langle X_t, \int_{\mathbb{R}^d} \widehat{f}(\lambda)\overline{\varphi_\lambda}\frac{d\lambda}{(2\pi)^d}\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\langle X_t, \overline{\varphi_\lambda}\rangle \widehat{f}(\lambda)d\lambda$. Denote by $W_t(\lambda) := e^{-\rho(\lambda)t}\langle X_t, \overline{\varphi_\lambda}\rangle$. Then we have the following lemma.

Lemma 3.1 *$\{W_t(\lambda), t \geq 0, \mathscr{G}_t, \mathbb{P}_{\delta_x}\}$ is a martingale for each $\lambda \in \mathbb{R}^d$. If $2\rho(\lambda) - \beta > 0$, $\{W_t(\lambda)\}$ converges almost surely and in the mean square.*

Proof. For each $\lambda \in \mathbb{R}^d$, by the Markov property and (2.3), we have

$$\begin{aligned}\mathbb{E}_{\delta_x}\left[W_{t+s}(\lambda)|\mathscr{G}_s\right] &= \mathbb{E}_{X_s}\left[e^{-(t+s)\rho(\lambda)}\langle X_t, \overline{\varphi_\lambda}\rangle\right] = e^{-(t+s)\rho(\lambda)}\langle P_t^\beta\overline{\varphi_\lambda}, X_s\rangle \\ &= e^{-s\rho(\lambda)}\langle X_s, \overline{\varphi_\lambda}\rangle = W_s(\lambda),\end{aligned}$$

so $\{W_t(\lambda), \mathscr{G}_t, \mathbb{P}_{\delta_x}\}$ is a martingale. By (2.2),

$$\begin{aligned}\mathbb{E}_{\delta_x}\left[|W_t(\lambda)|^2\right] &= e^{-2t\rho(\lambda)}\mathbb{E}_{\delta_x}\left[\langle X_t, \varphi_\lambda\rangle\langle X_t, \overline{\varphi_\lambda}\rangle\right] \\ &= e^{-2t\rho(\lambda)}e^{2\beta t}\left[(P_t\cos\lambda x)^2 + (P_t\sin\lambda x)^2\right] \\ &\quad +2\alpha e^{-2t\rho(\lambda)}\int_0^t ds\int_{\mathbb{R}^d}\left[(P_s^\beta\cos\lambda y)^2 + (P_s^\beta\sin\lambda y)^2\right]P_{t-s}^\beta(dy).\end{aligned}$$

Note that

$$\begin{aligned}P_t\cos\lambda x &= (2\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\cos\lambda y\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy \\ &= (2\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\cos\lambda(y-x+x)\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy \\ &= (2\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}[\cos\lambda(y-x)\cos\lambda x - \sin\lambda(y-x)\sin\lambda x]\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy \\ &= \frac{\cos\lambda x}{(\sqrt{2\pi t})^d}\int_{\mathbb{R}^d}\cos\lambda(y-x)\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy \\ &\quad - \frac{\sin\lambda x}{(\sqrt{2\pi t})^d}\int_{\mathbb{R}^d}\sin\lambda(y-x)\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy.\end{aligned}$$

Using the formula $\int_0^\infty e^{-x^2}\cos rx dx = \frac{\sqrt{\pi}}{2}e^{-\frac{r^2}{4}}$, we have

$$\begin{aligned}I_1 &:= \frac{\cos\lambda x}{(\sqrt{2\pi t})^d}\int_{\mathbb{R}^d}\cos\lambda(y-x)\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy \\ &= \frac{\cos\lambda x}{\sqrt{\pi}^d}\prod_{i=1}^d\left(\int_{\mathbb{R}}\cos\lambda_i\sqrt{2t}z_i e^{-z_i^2}dz_i\right) \\ &= \cos\lambda x e^{-\frac{t}{2}|\lambda|^2},\end{aligned}$$

where we have used the fact that "$\sin x$" is an odd function on the real line in the second equality. And similarly, we have

$$I_2 := \frac{\sin\lambda x}{(\sqrt{2\pi t})^d}\int_{\mathbb{R}^d}\sin\lambda(y-x)\exp\left\{-\frac{|y-x|^2}{2t}\right\}dy = 0.$$

Thus

$$P_t \cos \lambda x = \cos \lambda x e^{-\frac{t}{2}|\lambda|^2}.$$

Similarly,

$$\begin{aligned}
P_t \sin \lambda x &= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \sin \lambda y \exp\left\{-\frac{|y-x|^2}{2t}\right\} dy \\
&= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \sin \lambda (y-x+x) \exp\left\{-\frac{|y-x|^2}{2t}\right\} dy \\
&= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} [\sin \lambda (y-x) \cos \lambda x + \cos \lambda (y-x) \sin \lambda x] \exp\left\{-\frac{|y-x|^2}{2t}\right\} dy \\
&= \frac{\sin \lambda x}{(\sqrt{2\pi t})^d} \int_{\mathbb{R}^d} \cos \lambda (y-x) \exp\left\{-\frac{|y-x|^2}{2t}\right\} dy \\
&= \sin \lambda x e^{-\frac{t}{2}|\lambda|^2}.
\end{aligned}$$

Through the above calculation, we finally get,

$$\begin{aligned}
\mathbb{E}_{\delta_x}[|W_t(\lambda)|^2] &= 1 + 2\alpha e^{-(2\rho(\lambda)-\beta)t} \int_0^t e^{(2\rho(\lambda)-\beta)s} ds \\
&= \begin{cases} 1 + \frac{2\alpha}{2\rho(\lambda)-\beta}\left(1 - e^{-(2\rho(\lambda)-\beta)t}\right), & 2\rho(\lambda)-\beta \neq 0; \\ 1 + 2\alpha t, & 2\rho(\lambda)-\beta = 0. \end{cases}
\end{aligned} \tag{3.1}$$

Thus if $2\rho(\lambda) - \beta > 0$, then $0 < \sup_t \mathbb{E}_{\delta_x}[|W_t(\lambda)|^2] < \infty$. An application of the martingale convergence theorem implies that

$$W(\lambda) = \lim_{t\to\infty} W_t(\lambda) \tag{3.2}$$

exists almost surely and in the mean square. □

Denote by $\Lambda = \{\lambda : 2\rho(\lambda) - \beta > 0\}$. For every $\lambda \in \Lambda$, there is an exceptional null set N_λ. The fact that Λ is uncountable yields $\{W(\lambda); \lambda \in \Lambda\}$ can not be defined on the uncountable union $\bigcup_{\lambda\in\Lambda} N_\lambda$, so we have to prove the uniform convergence of $\{W_t(\lambda)\}$ on Λ or on some subset of Λ.

Definition 3.1 *Let $\mu \in M_c(\mathbb{R}^d)$. The measure valued process $\{X_t, t \geq 0\}$ with initial state μ possesses the compact support property if*

$$\mathbb{P}_\mu\left(\bigcup_{0\leq s\leq t} suppX(s) \Subset \mathbb{R}^d\right) = 1, \quad \textit{for all} \ \ t \geq 0.$$

Here the notation $A \Subset B$ means that A is bounded and $\bar{A} \subset B$.

For each $\mu \in M_c(\mathbb{R}^d)$, according to Theorem 3.4 and Theorem 3.5 of Engländer and Pinsky (1999), the corresponding super-Brownian motion with initial state μ possesses the compact support property. Thus we have the following lemma.

Lemma 3.2 *For each $\mu \in M_c(\mathbb{R}^d)$, $W_t(\lambda)$ is analytic in λ on $\mathbb{R}^d$ $\mathbb{P}_\mu$-almost surely, for all $t > 0$.*

Proof. Since $\{X_t, t \geq 0\}$ corresponding to $\mathbb{P}_\mu$ possesses the compact support property, then for each $t \geq 0$, $\int_{\mathbb{R}^d} |x| X_t(dx) < \infty$ almost surely. By dominated convergence theorem, $W_t(\lambda)$ is differentiable in λ almost surely. Then $W_t(\lambda)$ is differentiable in λ almost surely for all rational t. Thus $W_t(\lambda)$ is analytic in λ almost surely for all $t \geq 0$ by the right continuity of $W_\cdot(\lambda)$. $\square$

The following lemma was given in Biggins (1992). To state it, we first introduce some notations. The open polydisc centered at $\lambda_0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_d^0) \in \mathbb{R}^d$ with radius $\rho > 0$ is denoted by $D_{\lambda_0}(\rho)$ and defined by $D_{\lambda_0}(\rho) = \{\lambda \in \mathbb{R}^d : |\lambda_j - \lambda_j^0| < \rho, \forall j\}$, and its boundary $\Gamma_{\lambda_0}(\rho)$ is defined by $\Gamma_{\lambda_0}(\rho) = \{\lambda \in \mathbb{R}^d : |\lambda_j - \lambda_j^0| = \rho, \forall j\}$. Denote by

$$C = \{t \in \mathbb{R}^d : 0 \leq t_j \leq 2\pi, \forall j\} \quad \text{and} \quad \lambda_j(t) = \lambda_j^0 + 2\rho e^{it_j},$$

so that $\Gamma_{\lambda_0}(2\rho) = \{\lambda(t) : t \in C\}$.

Lemma 3.3 *If f is analytic on $D_{\lambda_0}(2\rho')$ with $\rho' > \rho$, then*

$$\sup_{\lambda \in D_{\lambda_0}(\rho)} |f(\lambda)| \leq \pi^{-d} \int_C |f(\lambda(t))| dt,$$

with C and $\lambda(t)$ as defined above.

For each $0 < \varepsilon < \beta$, denote by $\Lambda_\varepsilon := \{\lambda : 2\rho(\lambda) - \beta \geq \varepsilon\}$.

Proposition 3.1 *For every $\varepsilon > 0$ and every $x \in \mathbb{R}^d$, $\{W_t(\lambda)\}$ converges uniformly on Λ_ε, $\mathbb{P}_{\delta_x}$-almost surely, as $t \to \infty$.*

Proof. If $\lambda \in \Lambda_\varepsilon$, then the martingale property of $\{W_t(\lambda), t \geq 0\}$ and (3.1) imply that

$$\begin{aligned} \mathbb{E}_{\delta_x}[|W_{t+s}(\lambda) - W_t(\lambda)|^2] &\leq \mathbb{E}_{\delta_x}[|W_{t+s}(\lambda)|^2] - \mathbb{E}_{\delta_x}[|W_t(\lambda)|^2] \\ &= \frac{2\alpha}{2\rho(\lambda) - \beta}[1 - e^{-(2\rho(\lambda)-\beta)s}]e^{-(2\rho(\lambda)-\beta)t} \\ &\leq \frac{2\alpha}{\varepsilon} e^{-\varepsilon t}. \end{aligned} \tag{3.3}$$

Given any $\lambda_0 \in \Lambda_\varepsilon$, we can find $\rho > 0$ such that $D_{\lambda_0}(3\rho) \subset \Lambda_\varepsilon$. We use Lemma 3.3 to deduce that

$$\sup_{\lambda \in D_{\lambda_0}(\rho)} \pi^d |W_{t+s}(\lambda) - W_t(\lambda)| \leq \int_C |W_{t+s}(\lambda(u)) - W_t(\lambda(u))| du, \tag{3.4}$$

by (3.3) and Hölder's inequality,

$$\begin{aligned} \mathbb{E}_{\delta_x} \int_C |W_{t+s}(\lambda(u)) - W_t(\lambda(u))| du &\leq (2\pi)^d \sup_{\lambda \in \Gamma_{\lambda_0}(2\rho)} \mathbb{E}_{\delta_x}[|W_{t+s}(\lambda) - W_t(\lambda)|] \\ &\leq (2\pi)^d \sqrt{\frac{2\alpha}{\varepsilon}} e^{-\frac{\varepsilon}{2}t}, \end{aligned}$$

so the left side of (3.4) converges to zero almost surely as $t \to \infty$ and hence, by a compactness argument, we get the desired result. $\square$

Proposition 3.2 *For every $\varepsilon > 0$, $W(\lambda)$ is analytic on Λ_ε.*

Proof. As $W_t(\lambda)$ converges uniformly on Λ_ε to $W(\lambda)$ and $W_t(\lambda)$ is analytic in λ, standard complex analysis gives the analyticity of $W(\lambda)$, see Hörmander (1973), Corollary 2.2.4. □

The following theorems and corollaries are the main results of our paper. But first, it would be better to give a full statement of Lemma 3.4 in Watanabe (1967) which will be used below.

Lemma 3.4 *If Y is a non-negative random variable such that $P(Y > y) \le My^{-2}$, then for every $\eta > 0$,*

$$E(Y) \le \eta + M\eta^{-1}.$$

Proof.

$$E(Y) = -\int_0^\infty y dP(Y > y) = \int_0^\infty P(Y > y)dy \le \eta + M\int_\eta^\infty y^{-2}dy = \eta + M\eta^{-1}.$$

□

Theorem 3.1 *Assume $f \in \mathscr{A}$ and α, β are positive constants. Then for every ε such that $0 < \varepsilon < \frac{\beta}{2}$ and for every $x \in \mathbb{R}^d$, there exists $\delta > 0$ such that,*

$$\langle X_t, f\rangle = \frac{1}{(2\pi)^d}\int_{2\rho(\lambda)-\beta\ge\varepsilon} W(\lambda)e^{t\rho(\lambda)}\widehat{f}(\lambda)d\lambda + o(e^{(\beta-\delta)t}), \quad \mathbb{P}_{\delta_x} - a.s.$$

where $W(\lambda)$ is defined by (3.2).

Proof. If $f \in \mathscr{A}$, then $f(x) = \int_{\mathbb{R}^d}\widehat{f}(\lambda)\overline{\varphi_\lambda(x)}\frac{d\lambda}{(2\pi)^d}$. Hence,

$$\begin{aligned}\langle X_t, f\rangle &= \frac{1}{(2\pi)^d}\int_{2\rho(\lambda)-\beta\ge\varepsilon} W(\lambda)e^{t\rho(\lambda)}\widehat{f}(\lambda)d\lambda \\ &\quad + \frac{1}{(2\pi)^d}\int_{2\rho(\lambda)-\beta\ge\varepsilon}(W_t(\lambda)-W(\lambda))e^{t\rho(\lambda)}\widehat{f}(\lambda)d\lambda \\ &\quad + \frac{1}{(2\pi)^d}\int_{2\rho(\lambda)-\beta<\varepsilon} W_t(\lambda)e^{t\rho(\lambda)}\widehat{f}(\lambda)d\lambda \\ &:= \frac{1}{(2\pi)^d}(I_1(t)+I_2(t)+I_3(t)).\end{aligned}$$

First we shall show that

$$\mathbb{P}_{\delta_x}[\lim_{t\to\infty} e^{-(\beta-\delta)t}I_3(t) = 0] = 1. \tag{3.5}$$

Since $f \in \mathscr{A}$, there exists $c > 0$ such that $\int_{\mathbb{R}^d} e^{-2c\rho(\lambda)}|\widehat{f}(\lambda)|\frac{d\lambda}{(2\pi)^d} < \infty$. For every $y > 0$, we have by Doob's maximal inequality and (3.1) that

$$\begin{aligned}&\mathbb{P}_{\delta_x}\left(\sup_{cn\le t\le c(n+1)} e^{-(\frac{\beta}{2}+\varepsilon)t}e^{\rho(\lambda)t}|W_t(\lambda)| > y\right) \\ \le\ & \mathbb{P}_{\delta_x}\left(\sup_{cn\le t\le c(n+1)} |W_t(\lambda)| > ye^{-(\rho(\lambda)-\frac{\beta}{2}-\varepsilon)cn}\right) \\ \le\ & y^{-2}e^{2(\rho(\lambda)-\frac{\beta}{2}-\varepsilon)cn}\mathbb{E}_{\delta_x}\left[|W_{c(n+1)}(\lambda)|^2\right] \\ \le\ & \begin{cases} y^{-2}e^{2(\rho(\lambda)-\frac{\beta}{2}-\varepsilon)cn}\left(1+2c(n+1)\alpha\right), & 2\rho(\lambda)-\beta\ge 0; \\ y^{-2}e^{2(\rho(\lambda)-\frac{\beta}{2}-\varepsilon)cn}\left[1+2c(n+1)\alpha e^{-(2\rho(\lambda)-\beta)c(n+1)}\right], & 2\rho(\lambda)-\beta<0.\end{cases}\end{aligned}$$

Since $2\rho(\lambda)-\beta\le\varepsilon$, $2\rho(\lambda)-\beta-2\varepsilon<-\varepsilon$,

$$\mathbb{P}_{\delta_x}\left(\sup_{cn\le t\le c(n+1)} e^{-(\frac{\beta}{2}+\varepsilon)t}e^{\rho(\lambda)t}|W_t(\lambda)|>y\right)\le y^{-2}e^{-\varepsilon cn}\left[1+2c(n+1)\alpha\right]e^{-2(\rho(\lambda)-\beta)c}.$$

Using Lemma 3.4 by taking $\eta=e^{-\frac{cn\varepsilon}{2}}$, we get

$$\begin{aligned}&\mathbb{E}_{\delta_x}\left[\sup_{cn\le t\le c(n+1)} e^{-(\frac{\beta}{2}+\varepsilon-\rho(\lambda))t}|W_t(\lambda)|\right]\\ \le\ & e^{-\frac{cn\varepsilon}{2}}+[1+2c(n+1)\alpha]e^{-\frac{cn\varepsilon}{2}}e^{-2(\rho(\lambda)-\beta)c}\\ \le\ & C_1(n+1)e^{-\frac{cn\varepsilon}{2}}e^{-2\rho(\lambda)c}.\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}_{\delta_x}\left[\sup_{cn\le t\le c(n+1)} e^{-(\frac{\beta}{2}+\varepsilon)t}|I_3(t)|\right]\\ \le\ & \mathbb{E}_{\delta_x}\left[\int_{\mathbb{R}^d}\sup_{cn\le t\le c(n+1)} e^{-(\frac{\beta}{2}+\varepsilon-\rho(\lambda))t}|W_t(\lambda)||\widehat{f}(\lambda)|d\lambda\right]\\ \le\ & C_1(n+1)e^{-\frac{cn\varepsilon}{2}}\int_{\mathbb{R}^d}e^{-2c\rho(\lambda)}|\widehat{f}(\lambda)|d\lambda\\ \le\ & C_2(n+1)e^{-\frac{cn\varepsilon}{2}},\end{aligned}$$

where C_1, C_2 are constants. Hence

$$\mathbb{E}_{\delta_x}\left[\sum_n\sup_{cn\le t\le c(n+1)} e^{-(\frac{\beta}{2}+\varepsilon)t}|I_3(t)|\right]<\infty.$$

Choose δ such that $\beta-\delta\ge\frac{\beta}{2}+\varepsilon$, i.e., $\delta\le\frac{\beta}{2}-\varepsilon$, this proves (3.5).

Next we shall show that if $0<\delta<\frac{\varepsilon}{2}\wedge\beta$, we have

$$\mathbb{P}_{\delta_x}[\lim_{t\to\infty}e^{-(\beta-\delta)t}I_2(t)=0]=1. \tag{3.6}$$

For each $t\ge n$ and every $y>0$, by Doob's maximal inequality and (3.3),

$$\mathbb{P}_{\delta_x}\left(\sup_{n\le s\le t}|W_s(\lambda)-W_n(\lambda)|>y\right)\le y^{-2}\mathbb{E}_{\delta_x}[|W_t(\lambda)-W_n(\lambda)|^2]\le Ay^{-2}e^{-\varepsilon n}.$$

Applying dominated convergence theorem, we have

$$\mathbb{P}_{\delta_x}\left(\sup_{n\le t<\infty}|W_t(\lambda)-W_n(\lambda)|>y\right)\le Ay^{-2}e^{-\varepsilon n},$$

therefore

$$\mathbb{P}_{\delta_x}\left(|W(\lambda)-W_n(\lambda)|>y\right)\le Ay^{-2}e^{-\varepsilon n}.$$

Further we have

$$
\begin{aligned}
&\mathbb{P}_{\delta_x}\left(\sup_{n\le t<\infty}|W(\lambda)-W_t(\lambda)|>y\right)\\
\le\ & \mathbb{P}_{\delta_x}\left(\sup_{n\le t<\infty}|W_t(\lambda)-W_n(\lambda)|>\frac{y}{2}\right)+\mathbb{P}_{\delta_x}\left(|W(\lambda)-W_n(\lambda)|>\frac{y}{2}\right)\\
\le\ & A'y^{-2}e^{-\varepsilon n}.
\end{aligned}
$$

Using Lemma 3.4 in Watanabe (1967) by taking $\eta=e^{-\frac{\varepsilon}{2}n}$,

$$
\mathbb{E}_{\delta_x}[\sup_{n\le t<\infty}|W_t(\lambda)-W(\lambda)|]\le A''e^{-\frac{\varepsilon}{2}n}
$$

for some constant A'' independent of n. Then

$$
\begin{aligned}
&\mathbb{E}_{\delta_x}\left[\sup_{n\le t<n+1}e^{-(\beta-\delta)t}|I_2(t)|\right]\\
\le\ & \mathbb{E}_{\delta_x}\int_{2\rho(\lambda)-\beta\ge\varepsilon}\sup_{n\le t<n+1}\left(|W_t(\lambda)-W(\lambda)|e^{t(\rho(\lambda)-\beta+\delta)}\right)\widehat{f}(\lambda)d\lambda\\
\le\ & \mathbb{E}_{\delta_x}\int_{2\rho(\lambda)-\beta\ge\varepsilon}\sup_{n\le t<n+1}|W_t(\lambda)-W(\lambda)|e^{n(\rho(\lambda)-\beta)}e^{\delta(n+1)}\widehat{f}(\lambda)d\lambda\\
\le\ & A_1''e^{(\delta-\frac{\varepsilon}{2})n}\int_{2\rho(\lambda)-\beta\ge\varepsilon}e^{n(\rho(\lambda)-\beta)}\widehat{f}(\lambda)d\lambda\\
\le\ & A_2''e^{(\delta-\frac{\varepsilon}{2})n},
\end{aligned}
$$

where A_1'' and A_2'' are positive constants independent of n. Hence,

$$
\mathbb{E}_{\delta_x}\left[\sum_n\sup_{n\le t<n+1}e^{-(\beta-\delta)t}|I_2(t)|\right]<\infty.
$$

We get the desired result from (3.5) and (3.6) by taking $0<\delta<\frac{\varepsilon}{2}\wedge(\frac{\beta}{2}-\varepsilon)$. □

In the sequel we will frequently use the notation "$f(t)\sim g(t)$, $t\to\infty$", which means that $\lim_{t\to\infty}\frac{f(t)}{g(t)}=1$.

Theorem 3.2 *Assume α, β are positive constants. For every $x\in\mathbb{R}^d$ and every $f\in C_c(\mathbb{R}^d)$, we have*

$$
\lim_{t\to\infty}\frac{\langle X_t,f\rangle}{e^{\beta t}t^{-\frac{d}{2}}}=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}f(x)dx\cdot W(0),\quad \mathbb{P}_{\delta_x}-a.s.
$$

where $W(0)$ is the $\mathbb{P}_{\delta_x}$-almost sure limit of $e^{-\beta t}\langle X_t,1\rangle$.

Proof. We may set $\varepsilon<\frac{\beta}{3}$. If $f\in\mathscr{A}$, by Theorem 3.1, then as $t\to\infty$,

$$
\begin{aligned}
\frac{\langle X_t,f\rangle}{e^{(\beta-\delta)t}}\ &\sim\ \frac{1}{(2\pi)^d}\int_{2\rho(\lambda)-\beta>\varepsilon}W(\lambda)e^{t(\rho(\lambda)-\beta)}\cdot e^{\delta t}\widehat{f}(\lambda)d\lambda\\
&=\ \frac{1}{(2\pi)^d}\int_{\rho(\lambda)-\beta>-\varepsilon}W(\lambda)e^{t(\rho(\lambda)-\beta)}\cdot e^{\delta t}\widehat{f}(\lambda)d\lambda\\
&\quad+\frac{1}{(2\pi)^d}\int_{\frac{\varepsilon-\beta}{2}\le\rho(\lambda)-\beta\le-\varepsilon}W(\lambda)e^{t(\rho(\lambda)-\beta)}\cdot e^{\delta t}\widehat{f}(\lambda)d\lambda.
\end{aligned}
$$

By Proposition 3.2, $W(\lambda)$ is continuous on $\Lambda_\varepsilon := \{\lambda : 2\rho(\lambda) - \beta \geq \varepsilon\}$, hence

$$\frac{1}{(2\pi)^d} \int_{\frac{\varepsilon-\beta}{2} \leq \rho(\lambda)-\beta \leq -\varepsilon} W(\lambda) e^{t(\rho(\lambda)-\beta)} \cdot e^{\delta t} \widehat{f}(\lambda) d\lambda \to 0, \quad \text{as } t \to \infty.$$

Therefore

$$\frac{\langle X_t, f\rangle}{e^{(\beta-\delta)t}} \sim \frac{1}{(2\pi)^d} \int_{\rho(\lambda)-\beta > -\varepsilon} W(\lambda) e^{t(\rho(\lambda)-\beta)} \cdot e^{\delta t} \widehat{f}(\lambda) d\lambda, \quad \text{as } t \to \infty.$$

Since $W(\lambda)$ and $\widehat{f}(\lambda)$ are continuous in λ and $\rho(0) = \beta$, for sufficiently small ε, we have

$$\begin{aligned} \frac{\langle X_t, f\rangle}{e^{\beta t}} &\sim \frac{1}{(2\pi)^d} \int_{\beta-\rho(\lambda)<\varepsilon} e^{t(\rho(\lambda)-\beta)} d\lambda \cdot \int_{\mathbb{R}^d} f(x)dx \cdot W(0) \\ &\sim \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{t(\rho(\lambda)-\beta)} d\lambda \cdot \int_{\mathbb{R}^d} f(x)dx \cdot W(0), \quad \text{as } t \to \infty, \end{aligned}$$

when deducing the second $\sim$, we have used the fact that

$$\int_{\beta-\rho(\lambda)\geq\varepsilon} e^{t(\rho(\lambda)-\beta)} d\lambda = o\left(\int_{\mathbb{R}^d} e^{t(\rho(\lambda)-\beta)} d\lambda \right), \quad \text{as } t \to \infty.$$

Consequently, for $f \in \mathscr{A}$,

$$\lim_{t\to\infty} \frac{\langle X_t, f\rangle}{e^{\beta t} t^{-\frac{d}{2}}} = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)dx \cdot W(0).$$

An application of Lemma 2.1 gives the desired result for $f \in C_c(\mathbb{R}^d)$. $\square$

Remark In fact, Lemma 2.1 holds for every bounded Borel measurable function f on $\mathbb{R}^d$ whose set of discontinuous points has zero Lebesgue measure, so does Theorem 3.2. The Law of large numbers for super-Brownian motion had been proved in Engländer (2008), the following corollary gives the strong one.

Corollary 3.1 *(Stong law of large numbers) Assume α, β are positive constants. Then for every relatively compact Borel subset B in $\mathbb{R}^d$ with positive Lesbegue measure whose boundary has zero Lebesgue measure, we have $\mathbb{P}_{\delta_x}$-almost surely,*

$$\lim_{t\to\infty} \frac{X_t(B)}{\mathbb{P}_{\delta_x}[X_t(B)]} = W(0).$$

Proof. Note that

$$\begin{aligned} \mathbb{P}_{\delta_x}[X_t(B)] &= e^{\beta t} \int_{\mathbb{R}^d} 1_B(y) p_t(x,y) dy \\ &= (2\pi t)^{-\frac{d}{2}} e^{\beta t} \int_{\mathbb{R}^d} 1_B(y) \exp\left\{ -\frac{|y-x|^2}{2t} \right\} dy \\ &\sim (2\pi)^{-\frac{d}{2}} t^{-\frac{d}{2}} e^{\beta t} \int_{\mathbb{R}^d} 1_B(y) dy, \quad t \to \infty. \end{aligned}$$

Take $f(x) = 1_B(x)$ in Theorem 3.2,

$$\begin{aligned}\lim_{t\to\infty} \frac{X_t(B)}{\mathbb{P}_{\delta_x}[X_t(B)]} &= \lim_{t\to\infty} \frac{X_t(B)}{(2\pi)^{-\frac{d}{2}} t^{-\frac{d}{2}} e^{\beta t} \int_{\mathbb{R}^d} 1_B(y)dy} \\ &= W(0).\end{aligned}$$

□

For the case that α and β are spatially dependent functions in $C^\eta(\mathbb{R}^d)$, α is bounded and positive, and β is compactly supported, we have the following result.

Theorem 3.3 *Let $L = \frac{1}{2}\Delta$. For every $\beta \in C_c^\eta(\mathbb{R}^d)$, assume $\lambda_c = \lambda_c(\beta) > 0$. Then there exists $\Omega_0 \subset \Omega$ of probability one (that is $\mathbb{P}_{\delta_x}(\Omega_0) = 1$ for every $x \in \mathbb{R}^d$) such that, for every $\omega \in \Omega_0$ and every bounded measurable function f on $\mathbb{R}^d$ with compact support whose set of discontinuous points has zero Lebesgue measure, we have*

$$\lim_{t\to\infty} e^{-\lambda_c t}\langle X_t, f\rangle = M_\infty^\phi \int_{\mathbb{R}^d} f(x)\phi(x)dx.$$

where M_∞^ϕ is the almost sure limit of $e^{-\lambda_c t}\langle X_t, \phi\rangle$, ϕ is the normalized positive eigenfunction of $L + \beta$ corresponding to λ_c.

Remark This is an immediate consequence of Chen et al (2008). Note that, β is assumed to be compactly supported so it is a Green-tight function for Brownian motion, i.e., $\beta \in K_\infty(\xi)$, see Chung (1982, p128). For the definition of Green-tight function, the reader is refereed to Zhao (1993).

Under the assumption that $\lambda_c > 0$, the results of Chen et al (2008) imply immediately that the associated Schrodinger operator $L + \beta$ has a spectral gap and has an L^2-eigenfunction corresponding to λ_c, and the strong limit theorem holds.

4 Examples

In this section we will give an example to show that the main results also hold with some subdomains in place of $\mathbb{R}^d$.

Let $D = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d; x_{d-i+1} > 0, x_{d-i+2} > 0, \cdots, x_d > 0\}$. Let ξ be a Brownian motion on D with absorbing boundary and let X be a super-Markov process on D corresponding to the operator $\frac{1}{2}\Delta u + \beta u - \alpha u^2$ where α and β are positive constants. In this case, the transition density for ξ is

$$p_t(x,y) = (2\pi t)^{-\frac{d}{2}} \prod_{j=1}^{d-i} \left(\exp\left\{ -\frac{(y_j - x_j)^2}{2t} \right\} \right) \prod_{j=d-i+1}^{d} \left(\exp\left\{ -\frac{(y_j - x_j)^2}{2t} \right\} - \exp\left\{ -\frac{(y_j + x_j)^2}{2t} \right\} \right)$$

and we take

$$\varphi_\lambda(x) = \prod_{j=1}^{d-i} e^{i\lambda_j x_j} \prod_{j=d-i+1}^{d} \frac{\sin \lambda_j x_j}{\lambda_j},$$

$$\rho(\lambda) = \beta - \frac{1}{2}|\lambda|^2.$$

For $f \in C_c(D)$ and $\widehat{g} \in C_c(\mathbb{R}^d)$, define the generalized Fourier transform by

$$\widehat{f}(\lambda) = \int_D f(x)\varphi_\lambda(x)dx$$

and

$$g(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{g}(\lambda)\overline{\varphi_\lambda(x)} \left(\prod_{j=d-i+1}^{d} \lambda_j \right)^2 d\lambda.$$

For each $\mu \in M_c(D)$ (the space of finite measures with compact support on D), according to Theorem 1 of Engländer and Pinsky (2006) with D instead of $\mathbb{R}^d$, the superprocesses $\{X_t, t \geq 0\}$ corresponding to $\mathbb{P}_\mu$ possesses the compact support property. Thus, by similar cacaulations to that in section 3, we have for every $x \in D$,

$$\lim_{t\to\infty} \frac{\langle X_t, f\rangle}{e^{\beta t}t^{-\frac{d}{2}-i}} = (2\pi)^{-\frac{d}{2}} \int_D f(x) \left(\prod_{j=d-i+1}^{d} x_j \right) dx \cdot W(0), \quad \mathbb{P}_{\delta_x} - a.s.$$

where $W(0)$ is the $\mathbb{P}_{\delta_x}$-almost sure limit of $e^{-\beta t}\langle X_t, 1\rangle$.

Similarly it can be checked that Theorem 3.2 also applies to the examples in Watanabe (1967) with branching Brownian motion replaced by super-Brownian motion on some subdomains in $\mathbb{R}^d$ with absorbing boundary.

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